§3. Wess-Zumino-Witten model

In this paragraph we take our Lie group to be $G = SU(2)$.

The "Maurer-Cartan form" $\mu = X^{-1}dX$, $X \in SU(2)$, is a 1-form on G with values in the Lie algebra $su(2)$ of G.

Next, define the 3-form

$$\sigma = \mathrm{Tr}(\mu \wedge \mu \wedge \mu)$$

If we parametrize an element of $SU(2)$ by

$$X(\theta^1, \theta^2, \theta^3) = \exp\left(i \sum_j \theta^j t_j\right)$$

where t_j, $j = 1,2,3$ are the Pauli matrices, then we can write

$$\Gamma(X) := \int_{S^3} \sigma = \int d\theta^1 d\theta^2 d\theta^3 \, \varepsilon^{ijk} \, \mathrm{Tr}\left(X^{-1}\frac{\partial X}{\partial \theta^i} X^{-1}\frac{\partial X}{\partial \theta^j} \frac{\partial X}{\partial \theta^k}\right)$$

$\longrightarrow$ invariant under coordinate trfs. as

$$\varepsilon^{ijk} \frac{\partial \theta'^l}{\partial \theta^i} \frac{\partial \theta'^m}{\partial \theta^j} \frac{\partial \theta'^n}{\partial \theta^k} = \mathrm{Det}\left(\frac{\partial \theta'}{\partial \theta}\right) \varepsilon^{lmn}$$

Also invariant under small deformations

$$X \longmapsto X + \delta X$$

Proof:

$$\delta\Gamma(X) = 3\int d\theta^1 d\theta^2 d\theta^3 \; \varepsilon^{ijk}\, \mathrm{Tr}\left(X^{-1}\frac{\partial X}{\partial\theta^i}\, X^{-1}\frac{\partial X}{\partial\theta^j}\, \delta\left(X^{-1}\frac{\partial X}{\partial\theta^k}\right)\right) \qquad (*)$$

$$(\varepsilon^{ijk} = \varepsilon^{jki} = \varepsilon^{kij})$$

Now, the last factor in the trace is

$$\delta\left(X^{-1}\frac{\partial X}{\partial\theta^k}\right) = -X^{-1}\delta X\, X^{-1}\frac{\partial X}{\partial\theta^k} + X^{-1}\frac{\partial\,\delta X}{\partial\theta^k}$$

$$\left[\begin{aligned} \delta X^{-1} &= (X+\delta X)^{-1} - X^{-1} = (\mathbb{1} + X^{-1}\delta X)^{-1}X^{-1} - X^{-1} \\ &= \left(\exp(-\log(\mathbb{1}+X^{-1}\delta X)) - \mathbb{1}\right)X^{-1} = -X^{-1}\delta X\, X^{-1} + \mathcal{O}(\delta X^2) \end{aligned}\right.$$

$$= X^{-1}\underbrace{\frac{\partial}{\partial\theta^k}\left(\delta X\, X^{-1}\right)}\, X$$

$$\left[\begin{aligned} &= \frac{\partial(\delta X)}{\partial\theta^k}X^{-1} + \delta X\, \underbrace{\frac{\partial(X^{-1})}{\partial\theta^k}}_{= -X^{-1}\frac{\partial X}{\partial\theta^k}X^{-1}} \\ &= X^{-1}\frac{\partial(\delta X)}{\partial\theta^k} - X^{-1}\delta X\, X^{-1}\frac{\partial X}{\partial\theta^k} \end{aligned}\right.$$

Inserting into (*) and integrating by parts gives

$$\begin{aligned} \delta\Gamma(X) = -3\int d\theta^1 d\theta^2 d\theta^3 \varepsilon^{ijk}\,\mathrm{Tr}\Big(& X^{-1}\frac{\partial X}{\partial\theta^i}\left(\frac{\partial}{\partial\theta^k}X^{-1}\right)\frac{\partial X}{\partial\theta^j}\, X^{-1}\delta X \\ & + X^{-1}\frac{\partial X}{\partial\theta^i}\, X^{-1}\frac{\partial X}{\partial\theta^j}\left(\frac{\partial}{\partial\theta^k}X^{-1}\right)\delta X\Big) \end{aligned}$$

$$= 0$$

$\square$

$\longrightarrow \Gamma(X) = \Gamma(c)$ where c is the homotopy class to which $X(\theta)$ belongs.

The integrals $\Gamma(c)$ furnish a representation of the homotopy group $\pi_3(SU(2))$:

$$\Gamma(c_a \times c_b) = \Gamma(c_a) + \Gamma(c_b)$$

$c_a \times c_b$ consists of mappings equivalent to

$$X_{ab}(\theta) = \begin{cases} X_a(2\theta_1, \theta_2, \theta_3) & 0 \leq \theta_1 \leq \frac{1}{2} \\ X_b(2\theta_1 - 1, \theta_2, \theta_3) & \frac{1}{2} \leq \theta_1 \leq 1 \end{cases} .$$

$\rightarrow$ part of the integral $\Gamma(X_{ab})$ over $0 \leq \theta_1 \leq \frac{1}{2}$ and $\frac{1}{2} \leq \theta_1 \leq 1$ can be done by changing variables to $\theta_1' = 2\theta_1$ and $\theta_1' = 2\theta_1 - 1$

$\longrightarrow \Gamma(c_a) + \Gamma(c_b)$

In particular: $\Gamma(c^n) = n\Gamma(c)$

and $\pi_3(SU(2)) = \mathbb{Z}$

The result of performing $SU(2)$-trf with par. θ followed by one with par. φ is:

$$X(\varphi)\, X(\theta) = X(\theta'(\theta, \varphi)) \qquad (**)$$

$$\frac{\partial}{\partial \theta'} \longrightarrow X(\varphi) \frac{\partial X}{\partial \theta^l} \frac{\partial \theta^l}{\partial \theta'^i} = \frac{\partial X(\theta')}{\partial \theta'^i}$$

$\cdot(**)^{-1}$

$$\Longrightarrow \frac{\partial\theta^l}{\partial\theta'^i} X(\theta)^{-1} \frac{\partial X(\theta)}{\partial\theta^l} = X^{-1}(\theta') \frac{\partial X(\theta')}{\partial\theta'^i}$$

$\longrightarrow$ integrand $T(X)$ at point θ' is:

$$\varepsilon^{ijk} \mathrm{Tr}\left(X^{-1}(\theta') \frac{\partial X(\theta')}{\partial\theta'^i} X^{-1}(\theta') \frac{\partial X(\theta')}{\partial\theta'^j} X^{-1}(\theta') \frac{\partial X(\theta')}{\partial\theta'^k} \right)$$

$$= \mathrm{Det}\left(\frac{\partial\theta}{\partial\theta'}\right) \varepsilon^{lmn} \mathrm{Tr}\left(X(\theta)^{-1} \frac{\partial X(\theta)}{\partial\theta^l} X^{-1}(\theta) \frac{\partial X(\theta)}{\partial\theta^m} X^{-1}(\theta) \frac{\partial X(\theta)}{\partial X^n} \right)$$

Now, $SU(2)$ (like every Lie group) has a metric

$$\gamma_{ij}(\theta) = -\frac{1}{2} \mathrm{Tr}\left(X^{-1}(\theta) \frac{\partial X(\theta)}{\partial\theta^i} X^{-1}(\theta) \frac{\partial X(\theta)}{\partial\theta^j} \right)$$

$$= \delta_{ij} + \frac{\theta_i \theta_j}{1-\vec{\theta}^2}$$

Use

$$X(\theta) = \begin{pmatrix} \theta_4 + i\theta_3 & \theta_2 + i\theta_1 \\ -\theta_2 + i\theta_1 & \theta_4 - i\theta_3 \end{pmatrix} = \theta_4 + 2i\theta\cdot\vec{t}$$

where $\theta_4 = \pm\sqrt{1-\vec{\theta}^2}$ or $\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 = 1$ (S^3)

Then, $\gamma_{ij}(\theta)$ has the property

$$\gamma_{ij}(\theta') = \frac{\partial\theta^k}{\partial\theta'^i} \frac{\partial\theta^l}{\partial\theta'^j} \gamma_{kl}(\theta)$$

$$\longrightarrow \mathrm{Det}\left(\frac{\partial\theta}{\partial\theta'}\right) = \left(\frac{\mathrm{Det}\,\gamma(\theta')}{\mathrm{Det}\,\gamma(\theta)}\right)^{\frac{1}{2}}$$

$$\longrightarrow T(X) = \varepsilon^{ijk} \mathrm{Tr}\left(X^{-1}(\theta) \frac{\partial X(\theta)}{\partial\theta^i} X^{-1}(\theta) \frac{\partial X(\theta)}{\partial\theta^j} X^{-1}(\theta) \frac{\partial X(\theta)}{\partial\theta^k} \right)$$

$$\times \frac{1}{(\mathrm{Det}\,\gamma(\theta))^{\frac{1}{2}}} \int d^3\theta' \, (\mathrm{Det}\,\gamma(\theta'))^{\frac{1}{2}} .$$

Evaluate $\theta = 0$ by integrating over θ'

Using $X(\theta) \xrightarrow{\theta \to 0} \mathbb{1} + 2i\theta^i t_i$, we compute

$$\Gamma(X) = (2i)^3 \varepsilon^{ijk} \operatorname{Tr}(t_i t_j t_k)$$
$$\times \frac{1}{\sqrt{\operatorname{Det}(\gamma(0))}} \int d^3\theta' \sqrt{\operatorname{Det}\gamma(\theta')} .$$

Further, using

$$\operatorname{Det}(\gamma(\theta)) = \frac{1}{1-\vec{\theta}^2},$$

we get

$$\Gamma(X) = -8i\varepsilon^{ijk} \operatorname{Tr}(t_i t_j t_k) \int \frac{d^3\theta}{\sqrt{1-\vec{\theta}^2}},$$

and finally, using

$$4t_i t_j = \delta_{ij} + 2i\varepsilon^{ijl} t_l \text{ and } \operatorname{Tr}(t_l t_k) = \tfrac{1}{2}\delta_{lk},$$

we see that

$$8\varepsilon^{ijk} \operatorname{Tr}(t_i t_j t_k) = 2i\varepsilon^{ijk}\varepsilon^{ijk} = 12i,$$

$$\int \frac{d^3\theta}{\sqrt{1-\vec{\theta}^2}} = 2\int_0^1 \frac{4\pi r^2\, dr}{\sqrt{1-r^2}} = 2\pi^2.$$

(integral runs twice ($\theta_4 = \pm 1$) over interior of unit ball)

Altogether,

$$\Gamma(c^\nu) = 24\pi^2 \nu, \quad \nu \in \mathbb{Z} \text{ "winding number"}$$

From now on we take

$$T(X) = \int_{S^3} \sigma, \text{ where } \sigma = \frac{1}{24\pi^2} \mathrm{Tr}(\mu \wedge \mu \wedge \mu)$$

Then the above shows $\sigma \in H^3(SU(2), \mathbb{Z})$, i.e. σ is volume-form of $SU(2)$.

<u>Wess-Zumino-Witten action:</u>

Let Σ be a compact Riemann surface with $\partial\Sigma = \emptyset$.

Let $f : \Sigma \longrightarrow G$ be a smooth map

Define

$$E_\Sigma := -\sqrt{-1} \int_\Sigma \mathrm{Tr}\left(f^{-1}\partial f \wedge f^{-1}\bar{\partial} f\right)$$

as the "energy of f".

<u>Definition:</u>

The "Wess-Zumino-Witten" action $S_\Sigma(f)$ is defined by

$$S_\Sigma(f) = \frac{k}{4\pi} E_\Sigma(f) - \frac{\sqrt{-1}\,k}{12\pi} \int_B \mathrm{Tr}\left(\tilde{f}^{-1} d\tilde{f} \wedge \tilde{f}^{-1} d\tilde{f} \wedge \tilde{f}^{-1} d\tilde{f}\right)$$

where B is compact oriented smooth 3-manifold with $\partial B = \Sigma$ and $\tilde{f} : B \longrightarrow G$ s.t. $\tilde{f}|_\Sigma = f$.

$\longrightarrow \tilde{f} d\tilde{f}$ is pull-back of Maurer-Cartan form $\mu = X^{-1}dX$ by $\tilde{f}$.

<u>Lemma:</u>

$\exp(-S_\Sigma(f))$ does not depend on choice of B and extension $\tilde{f}$.

<u>Proof:</u>

Consider second 3-manifold B' with $\partial B' = \Sigma$ and $\tilde{f}': B \longrightarrow G$ s.t. $\tilde{f}'|_\Sigma = f$.

Define 3-manifold $M = B \cup|_\Sigma - B'$ where $-B'$ is B' with reverse orientation

Let: $F: M \longrightarrow G$ be a smooth map with $F|_B = \tilde{f}$ and $F|_{B'} = \tilde{f}'$. Then

$$\frac{k}{24\pi^2}\Big(\int_B \mathrm{Tr}(\tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f} \wedge \tilde{f}^{-1}d\tilde{f})$$

$$- \int_{B'} \mathrm{Tr}(\tilde{f}'^{-1}d\tilde{f}' \wedge \tilde{f}'^{-1}d\tilde{f}' \wedge \tilde{f}'^{-1}d\tilde{f}')$$

$$= k\int_M F^*\sigma \ , \quad \text{As } \sigma \in H^3(G,\mathbb{Z}) \longrightarrow \int_M F^*\sigma \in \mathbb{Z}$$

$$\Rightarrow \quad \exp\left(-\Delta S_\Sigma(f)\right) = \exp\Big(2\pi i\, k \underbrace{\int_M F^*\sigma}_{\in \mathbb{Z}}\Big) = 1$$

In other words, $\exp(-S_\Sigma(f))$ does not depend on choice of B and extensions $\tilde{f}$. $\square$

The term

$$\frac{\sqrt{-1}\,k}{12\pi}\int_B \mathrm{Tr}\left(\tilde{f}^{-1}d\tilde{f}\wedge\tilde{f}^{-1}d\tilde{f}\wedge\tilde{f}^{-1}d\tilde{f}\right)$$

is called "Wess-Zumino" term.